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Tensorial form definitions of discrete-mechanical quantities for granular assemblies

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Abstract

This paper explains the necessity of tensorial form definitions of mechanical quantities in the discrete mechanics of granular assemblies and how to make such definitions. Particles are assumed to be circular (in 2D) or spheres (in 3D). First, we explain the Dirichlet tessellation and some important geometric tools defined from this tessellation, i.e. the contact and dual-contact cells and dual branch vectors, which become necessary for the tensorial form definition. Comparing the fundamental quantities in discrete and continuum mechanics and regarding their mechanical properties, especially of stress and strain, we propose a new tensorial form definition for all of discrete-mechanical quantities. Lastly discussions are made on the properties of these quantities, especially on the internal work and the compatibility condition of strain.

Keywords: Granular assembly; Discrete mechanics; Stress and strain; Tensorial form definition; Compatibility condition

1. Introduction

In the micro-mechanics of granular assemblies, the stress is well defined using the contact forces and the branch vectors of an assembly's particle graph (e.g. Oda and Iwashita, 1999). However, the strain has not been clearly defined and its definition has been the subject of recent discussions (Bagi, 1996a; Kuhn, 1997; Satake, 2002; Kruyt, 2003). This paper proposes new and general definitions of stress, strain and other discrete-mechanical quantities of granular assemblies, which originate from vector quantities, in a tensorial form. We first introduce the Dirichlet tessellation (Dirichlet, 1850; also see Oda and Iwashita, 1999), which is a tessellation of space in a granular assembly. After explaining the correspondence between elements in the tessellation and the underlying granular assembly, we introduce some new geometric tools, which include contact and dual-contact cells and dual branch vectors. These cells and vectors appear in the tensorial form definitions of discrete-mechanical quantities of granular assemblies.

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Mechanical quantities in discrete mechanics of granular assemblies were originally defined in a vectorial form (Satake, 1993). However, definitions in a tensorial form are necessary for statistical analysis and for considering of bulk properties of a granular assembly. We propose a new definition of these discrete-mechanical quantities in a tensorial form, using the above-mentioned geometric tools. As the correspondence between discrete and continuum quantities is fundamentally important, definitions are introduced in the context of generalized continuum mechanics (Satake, 1971).

The equilibrium and compatibility conditions of stress and strain are compared for discrete and continuum systems. Lastly, we discuss some related problems with the proposed definition: Bagi's definition of strain, the form of internal work, and the compatibility conditions of the symmetric part of strain.

In this paper, particles are assumed to be spheres (in 3D) or disks (in 2D). The symbolic notation (in bold Roman and Greek letters without indices) is used for vector and tensor quantities and normal weight fonts are used for scalars.

2. Dirichlet tessellation

For a granular assembly, we introduce the so-called *Dirichlet tessellation*, as shown in a 2D form in Fig. 1. The Dirichlet tessellation is also known as the radical plane tessellation (Gellatly and Finney, 1982) and is a modification of the Voronoï tessellation (Voronoï, 1908). The graph of a Dirichlet tessellation consists of polygons (polyhedra in 3D), which we call *particle cells*. The dual graph of the Dirichlet tessellation is the Delaunay network, and the Delaunay network is a particle graph that is supplemented with virtual branches, which correspond to non-touching particles, as shown by the broken lines in Fig. 1. By adding these virtual branches, the Delaunay network becomes a non-overlapping covering of triangles (tetrahedra in 3D), which are called *void cell*s.

Next, we define the Dirichlet center of each void cell. In 2D, the three tessellating lines of the Dirichlet tessellation that are perpendicular to the three edges of a void cell (in 3D, the six tessellating planes that are perpendicular to the six edges of a void cell) always meet at a single point, as is shown in Fig. 2. This point is named the *Dirichlet center* of the void cell.

In 2D, a granular assembly has three elements; particles, contacts, and void cells, and in 3D, it has four elements; particles, contacts, dual-contacts and void cells. A *dual-contact* is a triangular face in the 3D Delaunay network through which two tetrahedral void cells are in contact. Table 1 shows the correspondence between elements in the graphs and elements in a 3D granular assembly. To specify the elements, we use the symbols P, C, D and $\overline{D}$. Originally, the symbols (running indices) denote particular spatial points

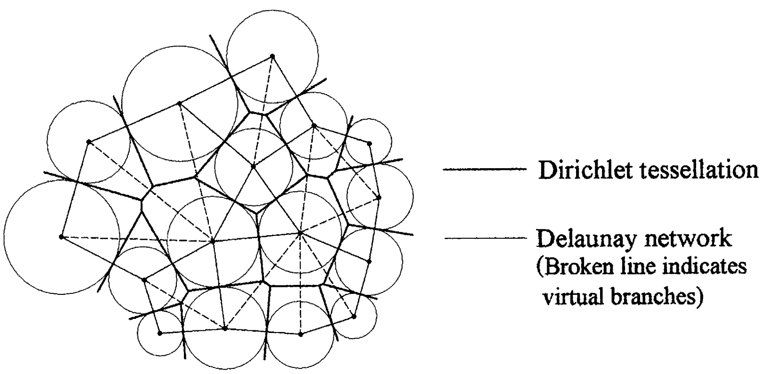
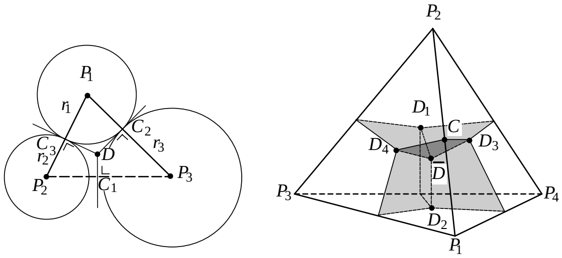


Fig. 1. Dirichlet tessellation (in 2D).



(a) 2D **(b)** 3D

Fig. 2. Dirichlet center.

Table 1
Correspondence between elements in graphs and a 3D granular assembly

Symbols	P (center of particle)	C (contact point)	D (Dirichlet center of dual-contact)	$\overline{D}$ (Dirichlet center of void cell)
Granular assembly	Particle	Contact	Dual-contact	Void cell
Dirichlet tessellation	Polyhedron	Polygon (face of polyhedron)	Edge of polygon	Node
Delaunay network	Node	Edge of triangle	Triangle (face of tetrahedron)	Tetrahedron
Particle graph	Point	Branch	Loop	Cell
Geometric quantities		Branch vector $\boldsymbol{l}_C$	Loop vector $\boldsymbol{S}_D$	
Connecting matrices	D_{PC}	L_{DC}	$C_{\overline{D}D}$	
Radius vectors	$\boldsymbol{r}_{PC}$	$\boldsymbol{s}_{DC}$	$\boldsymbol{t}_{\overline{D}D}$	

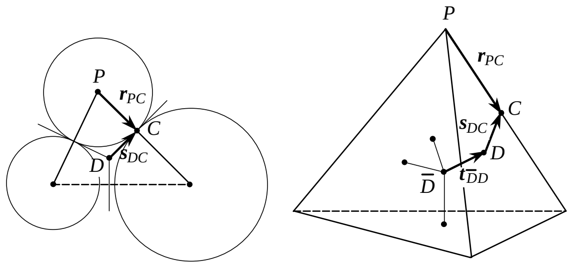
(as shown in the first line of Table 1). In 3D, P is a center of particle; C is a contact point (or virtual contact point); D is the Dirichlet center of a dual-contact (a triangular face of a tetrahedral void cell); and $\overline{D}$ is the Dirichlet center of a tetrahedral void cell. In 2D, P and C are the same as in 3D and D is the Dirichlet center of a void cell. In addition, the symbols are used in the following contexts:

- at times, we use the symbols to denote geometric objects associated with the above spatial points, as shown in Table 1. For example, the symbol C can refer to a branch of the assembly's particle graph and a polygonal face (an edge in 2D) of a particle cell, which are associated with the contact point C,
- a symbol can denote a region (area or volume) or mechanical quantities that are associated with the object. For example, a void cell $\overline{D}$ is a three-dimensional region of volume $V_{\overline{D}}$ with the Dirichlet center $\overline{D}$; the void strain γ_D (or $\gamma_{\overline{D}}$) is a discrete-mechanical strain defined for a void cell D in 2D (or $\overline{D}$ in 3D).

Denoting the position vectors of the Dirichlet centers D and $\overline{D}$ by $\boldsymbol{x}_D$ and $\boldsymbol{x}_{\overline{D}}$, respectively, we can give the following forms for a triangular void cell (or dual-contact in 3D) D and a tetrahedral void cell $\overline{D}$ (Satake, 2002):

$$\boldsymbol{x}_D = \frac{1}{2S_D} \sum_{i=1}^{3} c_i \widehat{\boldsymbol{l}}_i, \tag{1}$$

$$\boldsymbol{x}_{\overline{D}} = \frac{1}{3V_{\overline{D}}} \sum_{i=1}^{4} c_i \widehat{\boldsymbol{S}}_i. \tag{2}$$



(a) 2D **(b)** 3D

Fig. 3. Radius vectors.

$\widehat{\boldsymbol{l}}_C$: Vector associated with an edge C of the triangular void cell (dual-contact in 3D) D, whose magnitude is equal to the length of edge C and whose direction is the outward normal of edge C.

$\widehat{\boldsymbol{S}}_D$: Vector associated with a face D of the tetrahedral void cell $\overline{D}$, whose magnitude is equal to the area of face D and whose direction is the outward normal of face D.

S_D and $V_{\overline{D}}$ are the area of void cell (dual-contact in 3D) D and the volume of void cell $\overline{D}$, respectively, and

$$c_P = \tfrac{1}{2}(r_P^2 - \boldsymbol{x}_P \cdot \boldsymbol{x}_P), \tag{3}$$

where r_P is the radius of the particle P and $\boldsymbol{x}_P$ is the position vector of the center of particle P. The two vectors $\boldsymbol{x}_D$ and $\boldsymbol{x}_{\overline{D}}$ in Eqs. (1) and (2) share the same origin as the vector $\boldsymbol{x}_P$.

Three matrices, D_{PC}, L_{DC} and $C_{\overline{D}D}$ are used to express the topologic correspondence of the four elements (see Appendix). These matrices are composed of the elements 1, −1, and 0, which correspond to the oriented graphs of an assembly's topology. Signed values arise because of the directions of branches or the clockwise/counter clockwise orientations of triangular loops that are associated with the contacts between adjacent particles or the dual-contacts between adjacent void cells. These arbitrary directions or orientations must be assigned beforehand.

The *radius vector*s $\boldsymbol{r}_{PC}$, $\boldsymbol{s}_{DC}$ and $\boldsymbol{t}_{\overline{D}D}$ are shown in Fig. 3 and are used in the defining the geometric quantities in this paper. These radius vectors are defined by the locations of the spatial points P, C, D, and $\overline{D}$. For example, the radius vector $\boldsymbol{t}_{\overline{D}D}$ connects the Dirichlet center of a void cell $\overline{D}$ with the Dirichlet center of its face (dual-contact) D. Table 1 contains the connecting matrices and radius vectors described above.

3. Geometric description of contact cells and dual-contact cells

In a granular assembly, contact points are not distributed continuously but are discrete and scattered. A contact cell of finite area or volume becomes necessary for considering stress and strain that are defined at each contact point in the discrete mechanics of granular assemblies. That is, a region (volume) must be assigned to each contact (or dual-contact, as explained later), just as a volume is associated with each particle cell and void cell. The system of contact cells (or dual-contact cells) is a non-overlapping covering of the space of a granular assembly. The geometry of contact cells and dual-contact cells are described in this section.

A branch vector $\boldsymbol{l}_C$ is associated with each contact C, and it connects the centers of two particles of the contact. The *contact cell* of a contact C in 2D is a quadrangle formed by the branch vector $\boldsymbol{l}_C$ and the corresponding edge C (i.e. the contact edge D_1D_2 of the two neighboring particle cells P_1 and P_2), as is

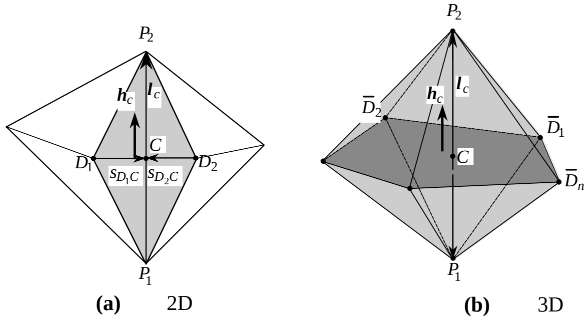


Fig. 4. Contact cell.

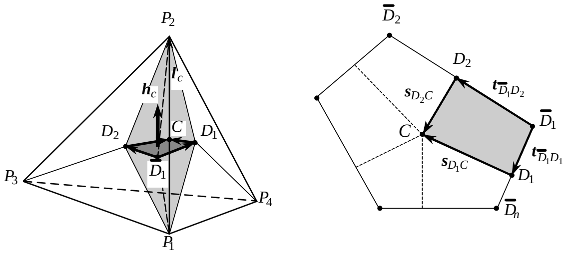


Fig. 5. A part of contact cell C that lies within a void cell $\overline{D}_1$.

shown in Fig. 4(a). We introduce a vector $\boldsymbol{h}_C$, expressed as the sum of two dual radius vectors $\hat{\boldsymbol{s}}_{D_1C}$ and $\hat{\boldsymbol{s}}_{D_2C}$ (see Fig. 4(a)) and given by the matrix sum

$$\boldsymbol{h}_C = L_{CD}(-\hat{\boldsymbol{s}}_{DC}). \tag{4}$$

Here we have introduced the "hat" notation "$\wedge$", which for a 2D vector $\boldsymbol{v} = (v_1, v_2)$ designates the rotated, dual vector $\hat{\boldsymbol{v}} = (v_2, -v_1)$. This notation will only be used in 2D analysis. The magnitude of $\boldsymbol{h}_C$ is equal to the length of edge C ($\overline{D_1D_2}$), and its direction is the same as that of $\boldsymbol{l}_C$. Note that the direction of $\boldsymbol{l}_C$ is chosen arbitrarily beforehand, and matrix L_{CD} is the transpose of the loop matrix L_{DC} (see Appendix). $\boldsymbol{h}_C$ is called the *dual branch vector* of C, and has an important role in the definition of discrete-mechanical strain, as is explained later. The area of contact cell C is written as

$$S_C = \tfrac{1}{2}\,\boldsymbol{l}_C \cdot \boldsymbol{h}_C. \tag{5}$$

In 3D, the contact cell is a polyhedron formed by a branch vector $\boldsymbol{l}_C$ and the corresponding face C (i.e. the planar contact face $\overline{D_1D_2}\cdots\overline{D_n}$ of two neighboring particle cells (polyhedrons) P_1 and P_2), as is shown in Fig. 4(b). Fig. 5 shows the part of a contact cell C that lies within a single void cell (tetrahedron) $\overline{D_1}$. Regarding Figs. 4(b) and 5, the dual branch vector in 3D is defined as the matrix product

$$\boldsymbol{h}_C = L_{CD}C_{D\overline{D}}\big(\tfrac{1}{2}\,\boldsymbol{t}_{\overline{D}D} \times \boldsymbol{s}_{DC}\big), \tag{6}$$

where $\boldsymbol{t}_{\overline{D}D}$ is the radius vector from the Dirichlet center $\overline{D}$ (of the void cell $\overline{D}$) to the Dirichlet center D (of the dual-contact D) (Figs. 3(b) and 5). The magnitude of $\boldsymbol{h}_C$ is equal to the area of face $C(\overline{D_1D_2}\cdots\overline{D_n})$, and the direction is the same as that of $\boldsymbol{l}_C$. The volume of contact cell C is written as

$$V_C = \tfrac{1}{3}\,\boldsymbol{l}_C \cdot \boldsymbol{h}_C. \tag{7}$$

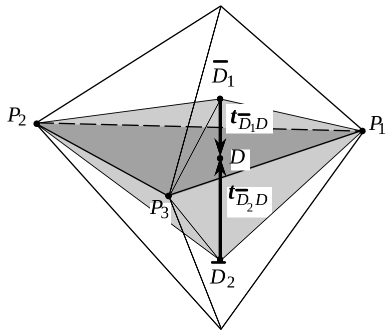


Fig. 6. Dual-contact cell.

We now describe the *dual-contact cell* in 3D. Consider two neighboring void cells (tetrahedra) $\overline{D_1}$ and $\overline{D_2}$ that share the dual-contact (triangular face) D, as is shown in Fig. 6. The dual-contact cell D is formed from the dual-contact (triangle) D and the two Dirichlet centers $\overline{D_1}$ and $\overline{D_2}$. This dual-contact cell will be used in the definition of the discrete-mechanical stress function tensors $(\varphi_D, \chi_D)^{\mathrm{T}}$ and the incompatibility tensors $(\boldsymbol{J}_D, \boldsymbol{K}_D)^{\mathrm{T}}$, as is explained in Section 6. Here we define the two vectors expressed as

$$\boldsymbol{S}_D = L_{DC}\left(\tfrac{1}{2}\,\boldsymbol{s}_{DC} \times \boldsymbol{l}_C\right), \tag{8}$$

$$\boldsymbol{k}_D = C_{D\overline{D}}\boldsymbol{t}_{\overline{D}D}, \tag{9}$$

and the volume of dual-contact cell D is written as

$$V_D = \tfrac{1}{3}\,\boldsymbol{S}_D \cdot \boldsymbol{k}_D. \tag{10}$$

$\boldsymbol{S}_D$ and $\boldsymbol{k}_D$ are called the *loop vector* and *dual loop vector* of dual-contact D, respectively, and the magnitude of $\boldsymbol{S}_D$ is equal to the area of triangular loop D, and the orientation of $\boldsymbol{S}_D$ (clockwise or counter-clockwise) is the same as that of loop D.

4. Mechanical quantities in discrete and continuum mechanics

In the 3D discrete mechanics of granular assemblies (Satake, 1997a), four mechanical quantities—force, couple, displacement and rotation—are associated with each of the four elements of a granular assembly (particles, contacts, dual-contacts and void cells), for a total of 16 mechanical quantities, as is shown in Table 2. The contact displacement $\boldsymbol{u}_C$ and rotation $\boldsymbol{w}_C$ mean the relative displacement and rotation at a contact point, respectively. The quantities of contacts and dual-contacts are defined for the positively connected particle and void cell, respectively, where a positive connection means that the corresponding component of matrix D_{PC} or $C_{\overline{D}D}$ is equal to 1 (see Appendix). The meanings of mechanical quantities for dual-contacts and void cells will be explained later. In 3D, all of the mechanical quantities are vectors. In 2D, we have only the first three mechanical quantities in Table 2, in which couples and rotations are scalars.

In generalized continuum mechanics (Satake, 1971), we can define each of the four mechanical quantities in 3D in four different forms, as is shown in Table 3. In 3D, $\boldsymbol{F}$, $\boldsymbol{M}$, $\boldsymbol{X}$, $\boldsymbol{Y}$ are defined in a volume element; σ, μ, $\boldsymbol{J}$, $\boldsymbol{K}$ on an area element; φ, χ, α, γ along a line element; and $\boldsymbol{f}$, $\boldsymbol{m}$, $\boldsymbol{w}$, $\boldsymbol{u}$ at a point. In 2D, $\boldsymbol{F}$, M, J,$\boldsymbol{K}$ are defined on an area element; σ, μ, α, γ along a line element; and φ, χ, w, u at a point. In common terminology, $\boldsymbol{F}$, σ, and φ are body force, stress, and stress function; and γ and $\boldsymbol{K}$ are strain and incompatibility, respectively. It is noted that, in 3D, the quantities of the second and third columns in Table 3 are tensors and those of the first and fourth columns are vectors. And, in 2D, the quantities become first three kinds, and M, w, χ, J become scalars and μ, α, φ, $\boldsymbol{K}$ vectors.

Table 2
Discrete-mechanical quantities for granular assemblies

	Particle	Contact	Dual-contact (Void cell) 2D	Void cell 3D
Force	$\boldsymbol{f}_P$	$\boldsymbol{f}_C$	$\boldsymbol{f}_D$	$\boldsymbol{f}_{\overline{D}}$
Couple	$\boldsymbol{m}_P$	$\boldsymbol{m}_C$	$\boldsymbol{m}_D$	$\boldsymbol{m}_{\overline{D}}$
Displacement	$\boldsymbol{u}_P$	$\boldsymbol{u}_C$	$\boldsymbol{u}_D$	$\boldsymbol{u}_{\overline{D}}$
Rotation	$\boldsymbol{w}_P$	$\boldsymbol{w}_C$	$\boldsymbol{w}_D$	$\boldsymbol{w}_{\overline{D}}$

Table 3
Mechanical quantities in generalized continuum mechanics

			2D	3D
Force	$\boldsymbol{F}$	σ	φ	$\boldsymbol{f}$
Couple	$\boldsymbol{M}$	μ	χ	$\boldsymbol{m}$
Rotation	$\boldsymbol{w}$	α	$\boldsymbol{J}$	$\boldsymbol{X}$
Displacement	$\boldsymbol{u}$	γ	$\boldsymbol{K}$	$\boldsymbol{Y}$

For quantities in generalized continuum mechanics, we can introduce some generating equations as is shown in Table 4, where *Grad*, *Rot*, *Div* are called the Schaefer's differential operators (Schaefer, 1967), defined as shown in Table 5. Note that the following identities hold for continuous fields:

$$Div\ Rot = 0, \quad Rot\ Grad = 0. \tag{11}$$

For mechanical quantities in the discrete mechanics of granular assemblies, we have the quite similar generating equations shown in Table 6 (Satake, 1993, 1997a). The operators in Table 6 are the following matrices and are called the fundamental matrices of the discrete mechanics of granular assemblies:

$$\begin{aligned}
\widetilde{D}_{PC} &= \begin{pmatrix} D_{PC} & 0 \\ D_{PC}\boldsymbol{r}_{PC}\times & D_{PC} \end{pmatrix}, &\quad \widetilde{D}_{CP} &= \begin{pmatrix} D_{CP} & -D_{CP}\boldsymbol{r}_{PC}\times \\ 0 & D_{CP} \end{pmatrix}, \\
\widetilde{L}_{CD} &= \begin{pmatrix} L_{CD} & 0 \\ -L_{CD}\boldsymbol{s}_{DC}\times & L_{CD} \end{pmatrix}, &\quad \widetilde{L}_{DC} &= \begin{pmatrix} L_{DC} & L_{DC}\boldsymbol{s}_{DC}\times \\ 0 & L_{DC} \end{pmatrix}, \\
\widetilde{C}_{D\overline{D}} &= \begin{pmatrix} C_{D\overline{D}} & 0 \\ -C_{D\overline{D}}\boldsymbol{t}_{\overline{D}D}\times & C_{D\overline{D}} \end{pmatrix}, &\quad \widetilde{C}_{\overline{D}D} &= \begin{pmatrix} C_{\overline{D}D} & C_{\overline{D}D}\boldsymbol{t}_{\overline{D}D}\times \\ 0 & C_{\overline{D}D} \end{pmatrix}.
\end{aligned} \tag{12}$$

Table 4
Generating relations in generalized continuum mechanics

Force *Couple*	$\begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{M} \end{pmatrix} = Div \begin{pmatrix} \sigma \\ \mu \end{pmatrix}$,	$\begin{pmatrix} \sigma \\ \mu \end{pmatrix} = Rot \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$,	$\begin{pmatrix} \varphi \\ \chi \end{pmatrix}$ (2D) $= Grad \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{m} \end{pmatrix}$,	$\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{m} \end{pmatrix}$ (3D)
Rotation *Displacement*	$\begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{u} \end{pmatrix}$,	$\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = Grad \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{u} \end{pmatrix}$,	$\begin{pmatrix} \boldsymbol{J} \\ \boldsymbol{K} \end{pmatrix} = Rot \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}$, (2D)	$\begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{pmatrix} = Div \begin{pmatrix} \boldsymbol{J} \\ \boldsymbol{K} \end{pmatrix}$ (3D)

Table 5
Schaefer's differential operators

	Grad	Rot	Div
2D	$\begin{pmatrix} \nabla & 0 \\ -\widehat{\boldsymbol{I}} & \nabla \end{pmatrix}$	$\begin{pmatrix} \nabla\times & 0 \\ -\boldsymbol{I}\cdot & \nabla\times \end{pmatrix}$	
2D		$\begin{pmatrix} \widehat{\nabla} & 0 \\ -\boldsymbol{I}\cdot & \nabla \end{pmatrix}$	$\begin{pmatrix} \nabla\cdot & 0 \\ \boldsymbol{I}\cdot\times & \nabla\cdot \end{pmatrix}$
3D	$\begin{pmatrix} \nabla & 0 \\ \boldsymbol{I}\times & \nabla \end{pmatrix}$	$\begin{pmatrix} \nabla\times & 0 \\ \boldsymbol{I}\times\times & \nabla\times \end{pmatrix}$	$\begin{pmatrix} \nabla\cdot & 0 \\ \boldsymbol{I}\cdot\times & \nabla\cdot \end{pmatrix}$

$\widehat{\nabla} = (\partial_{x_2}, -\partial_{x_1})$, $\widehat{\boldsymbol{I}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Table 6
Generating relations in discrete mechanics

	Particle	*Contact*	*Dual-contact (Void cell)*	*Void cell*
Force *Couple*	$\begin{pmatrix} \boldsymbol{f}_P \\ \boldsymbol{m}_P \end{pmatrix} = -\tilde{D}_{PC} \begin{pmatrix} \boldsymbol{f}_C \\ \boldsymbol{m}_C \end{pmatrix}$,	$\begin{pmatrix} \boldsymbol{f}_C \\ \boldsymbol{m}_C \end{pmatrix} = \tilde{L}_{CD} \begin{pmatrix} \boldsymbol{f}_D \\ \boldsymbol{m}_D \end{pmatrix}$,	$\begin{pmatrix} \boldsymbol{f}_D \\ \boldsymbol{m}_D \end{pmatrix}$ (2D) $= \tilde{C}_{D\overline{D}} \begin{pmatrix} \boldsymbol{f}_{\overline{D}} \\ \boldsymbol{m}_{\overline{D}} \end{pmatrix}$,	$\begin{pmatrix} \boldsymbol{f}_{\overline{D}} \\ \boldsymbol{m}_{\overline{D}} \end{pmatrix}$ (3D)
Displacement *Rotation*	$\begin{pmatrix} \boldsymbol{u}_P \\ \boldsymbol{w}_P \end{pmatrix}$,	$\begin{pmatrix} \boldsymbol{u}_C \\ \boldsymbol{w}_C \end{pmatrix} = -\tilde{D}_{CP} \begin{pmatrix} \boldsymbol{u}_P \\ \boldsymbol{w}_P \end{pmatrix}$,	$\begin{pmatrix} \boldsymbol{u}_D \\ \boldsymbol{w}_D \end{pmatrix} = \tilde{L}_{DC} \begin{pmatrix} \boldsymbol{u}_C \\ \boldsymbol{w}_C \end{pmatrix}$, (2D)	$\begin{pmatrix} \boldsymbol{u}_{\overline{D}} \\ \boldsymbol{w}_{\overline{D}} \end{pmatrix} = \tilde{C}_{\overline{D}D} \begin{pmatrix} \boldsymbol{u}_D \\ \boldsymbol{w}_D \end{pmatrix}$ (3D)

Using Eq. (A.4) in Appendix, we have the following identities (Satake, 1997b):

$$\left.\begin{array}{ll} \widetilde{D}_{PC}\widetilde{L}_{CD} = 0, & \widetilde{L}_{CD}\widetilde{C}_{D\overline{D}} = 0, \\ \widetilde{C}_{\overline{D}D}\widetilde{L}_{DC} = 0, & \widetilde{L}_{DC}\widetilde{D}_{CP} = 0, \end{array}\right\} \tag{13}$$

which are quite analogous to Eq. (A.4) and correspond to Eq. (11) in continuum mechanics. In 2D, the operators $-D_{CP}\boldsymbol{r}_{PC}\times$ and $L_{DC}\boldsymbol{s}_{DC}\times$ in Eq. (12) should be replaced by $-D_{CP}\widehat{\boldsymbol{r}}_{PC}$ and $L_{DC}\widehat{\boldsymbol{s}}_{DC}$, respectively.

5. Discrete-mechanical definition of stress and strain

In this section, we review the definition of stress both in continuum and discrete mechanics and introduce the definition of strain along the same line of consideration.

The stress in 3D continuum mechanics is defined, for a small sphere with radius a, as

$$\sigma = \lim_{a\to 0}\frac{1}{V}\oint \boldsymbol{rf}\,\mathrm{d}S, \tag{14}$$

where $\boldsymbol{r}$ and $\boldsymbol{f}$ denote the radius vector and stress vector, respectively, and V is the volume of the sphere. Note that $\boldsymbol{rf}$ denotes a dyadic product of $\boldsymbol{r}$ and $\boldsymbol{f}$, which can also be written as $\boldsymbol{r}\otimes\boldsymbol{f}$.

In a quite similar manner, in discrete mechanics, we can define the *particle stress*, the stress for a particle P, in the following form:

$$\sigma_P = \frac{1}{V_P}D_{PC}\boldsymbol{r}_{PC}\boldsymbol{f}_C, \tag{15}$$

where $\boldsymbol{r}_{PC}$ is the radius vector from the center of particle P to the contact point C, and V_P is the volume of the particle cell (Fig. 7). When D_{PC} is −1, contact force $\boldsymbol{f}_C$ acts upon particle P; when D_{PC} is −1, $-\boldsymbol{f}_C$ acts upon particle P. Note that, in the more general case, this definition is applicable even to non-spherical particles. For an RVE (representative volume element) R that includes a sufficient number of contiguous particles, we can define the global stress σ, an average stress having the following forms:

$$\sigma = \frac{1}{V}\sum_P V_P\sigma_P = \frac{1}{V}\sum_C \boldsymbol{l}_C\boldsymbol{f}_C, \tag{16}$$

where V is the volume of R and branch vector

$$\boldsymbol{l}_C = D_{CP}\boldsymbol{r}_{PC} \tag{17}$$

with D_{CP} denoting the transpose of the incidence matrix D_{PC}. Here $\sum_P$ or $\sum_C$ means that the summation should be made for $P\in R$ or $C\in R$. At the boundary of R, Eq. (17) is not strictly satisfied for $P\in R$ or

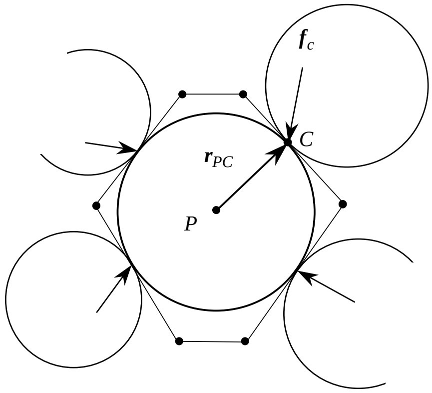


Fig. 7. Particle cell P for definition of particle stress σ_P.

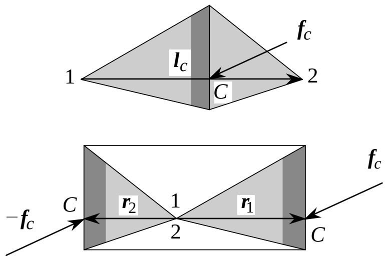


Fig. 8. Area element for definition of contact stress σ_C.

$C \in R$. However, as the effect is considered as small, the error in Eq. (16) due to this matter will be disregarded. It is noteworthy that Eq. (16) is also derivable from the virtual work principle (Christoffersen et al., 1981).

We also introduce the *contact stress* of a contact C, defined as

$$\sigma_C = \frac{1}{3V_C} \boldsymbol{l}_C \boldsymbol{f}_C. \tag{18}$$

The contact stress is associated with an element of volume $3V_C$ (area $2S_C$ in 2D, see Fig. 8) and is an incomplete tensor, as understood from the definition. From Eq. (18) we find the global stress is written as

$$\sigma = \frac{1}{V} \sum_C 3V_C \sigma_C. \tag{19}$$

For the couple stress, we can also derive similar definitions of μ_P,μ, and μ_C and their related equations as explained for stress, replacing $\boldsymbol{f}_C$ with the contact couple $\boldsymbol{m}_C$ in Eqs. (15)–(19).

We now proceed to the strain γ. Referring to a relation in generalized continuum mechanics, expressed as

$$\mathrm{d}\boldsymbol{x} \cdot \gamma = \mathrm{d}\boldsymbol{u} + \mathrm{d}\boldsymbol{x} \times \boldsymbol{w}, \tag{20}$$

where $\boldsymbol{u}$ and $\boldsymbol{w}$ are a displacement and rotation (Table 3), we can write, for a small circle of radius a,

$$\gamma = \lim_{a \to 0} \frac{1}{\pi a} \oint \boldsymbol{t}(\mathrm{d}\boldsymbol{u} + \mathrm{d}\boldsymbol{x} \times \boldsymbol{w}), \tag{21}$$

where $\boldsymbol{t}$ is a unit tangent vector with the same direction as $\mathrm{d}\boldsymbol{x}$.

Regarding Eq. (21), we define the *void strain* (in 2D), the strain of a void cell D (Fig. 9) in the discrete mechanics of granular material:

$$\gamma_D = \frac{1}{S_D} L_{DC}(-\widehat{\boldsymbol{s}}_{DC}) \boldsymbol{u}_C \tag{22}$$

recalling the "$\wedge$" notation defined after Eq. (4). In Eq. (22), $\boldsymbol{s}_{DC}$ and $\boldsymbol{u}_C$ are the void cell radius vector (Fig. 3) and the contact displacement (Table 2), respectively, and

$$S_D = \tfrac{1}{2} L_{DC}(-\widehat{\boldsymbol{s}}_{DC}) \cdot \boldsymbol{l}_C = \tfrac{1}{2} L_{DC} \boldsymbol{s}_{DC} \times \boldsymbol{l}_C \tag{23}$$

is the area of the void cell D. For a two-dimensional RVE with area S, the average strain is defined in the following form:

$$\gamma = \frac{1}{S} \sum_D S_D \gamma_D = \frac{1}{S} \sum_D L_{DC}(-\widehat{\boldsymbol{s}}_{DC}) \boldsymbol{u}_C = \frac{1}{S} \sum_C L_{CD}(-\widehat{\boldsymbol{s}}_{DC}) \boldsymbol{u}_C = \frac{1}{S} \sum_C \boldsymbol{h}_C \boldsymbol{u}_C. \tag{24}$$

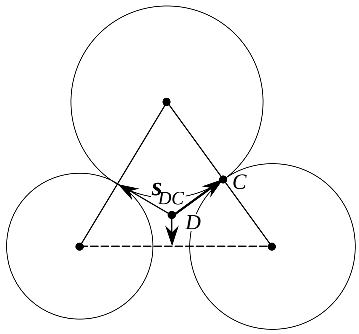


Fig. 9. Void cell D for definition of void strain γ_D.

In 3D, we define the void strain, the strain for a void cell $\overline{D}$, in the following form:

$$\gamma_{\overline{D}} = \frac{1}{V_{\overline{D}}} C_{\overline{D}D} L_{DC} \left(\frac{1}{2} \boldsymbol{t}_{\overline{D}D} \times \boldsymbol{s}_{DC} \right) \boldsymbol{u}_C, \tag{25}$$

where $\boldsymbol{t}_{\overline{D}D}$ and $\boldsymbol{s}_{DC}$ are radius vectors (Figs. 3(b) and 5), and

$$V_{\overline{D}} = C_{\overline{D}D} \tfrac{1}{3} \boldsymbol{t}_{\overline{D}D} \cdot L_{DC} \tfrac{1}{2} \boldsymbol{s}_{DC} \times \boldsymbol{l}_C = \tfrac{1}{6} C_{\overline{D}D} L_{DC} \boldsymbol{l}_C \cdot (\boldsymbol{t}_{\overline{D}D} \times \boldsymbol{s}_{DC}) \tag{26}$$

is the volume of the void cell $\overline{D}$. The meaning of strain $\gamma_{\overline{D}}$ will be described below. The global strain for an RVE of volume V is defined in the following forms:

$$\gamma = \frac{1}{V} \sum_{\overline{D}} V_{\overline{D}} \gamma_{\overline{D}} = \frac{1}{V} \sum_{\overline{D}} C_{\overline{D}D} L_{DC} \left(\frac{1}{2} \boldsymbol{t}_{\overline{D}D} \times \boldsymbol{s}_{DC} \right) \boldsymbol{u}_C = \frac{1}{V} \sum_{C} L_{CD} C_{D\overline{D}} \left(\frac{1}{2} \boldsymbol{t}_{\overline{D}D} \times \boldsymbol{s}_{DC} \right) \boldsymbol{u}_C = \frac{1}{V} \sum_{C} \boldsymbol{h}_C \boldsymbol{u}_C. \tag{27}$$

In this equation, we have used the identity (6), and, by doing so, have expressed the global strain in alternative forms: as a sum over void cell volumes and as a sum over contact cell volumes (see Eq. (30)).

As the contact displacement $\boldsymbol{u}_C$ includes the effect of particle rotations, referring to Eq. (20), we introduce the *contact strain* γ_C, the strain at a contact C, having the property

$$\boldsymbol{l}_C \cdot \gamma_C = \boldsymbol{u}_C. \tag{28}$$

Using Eq. (7), we can write the following definition of strain γ_C, which satisfies Eq. (28):

$$\gamma_C = \frac{1}{3V_C} \boldsymbol{h}_C \boldsymbol{u}_C, \tag{29}$$

noting, however, that Eq. (28) does not infer a unique definition of contact strain γ_C, such as the one given in Eq. (29). The global strain γ defined from $\gamma_{\overline{D}}$ is also written as

$$\gamma = \frac{1}{V} \sum_{C} 3 V_C \gamma_C. \tag{30}$$

It is noted here that, like as the contact stress σ_C, the contact strain γ_C is an incomplete tensor. From this reason, the void strain $\gamma_{\overline{D}}$ (or γ_D in 2D) that is a complete tensor becomes necessary in the discrete mechanics of granular assemblies. By adopting Eq. (29) as a generating definition of contact strain γ_C, we can now explain the meaning of the void strain $\gamma_{\overline{D}}$ in Eq. (25). Unlike other definitions of strain in granular materials (e.g., Bagi, 1996a; Kruyt and Rothenburg, 1996; Kuhn, 1997), the void strain $\gamma_{\overline{D}}$ does not represent the average deformation of a tetrahedron whose vertices (particle centers) are being displaced but whose edges remain straight. Such strain definitions are based only on the displacements $\boldsymbol{u}_P$ of the particle centers; whereas, $\gamma_{\overline{D}}$ is based upon the contact displacements $\boldsymbol{u}_C$, which depend both upon the particle displacements and rotations, $\boldsymbol{u}_P$ and $\boldsymbol{w}_P$, as given in Table 6. The contact strain γ_C is generated from the contact displacement $\boldsymbol{u}_C$ (see Eq. (28)), and the void strain $\gamma_{\overline{D}}$ in Eq. (25) is inferred from the identical sums in Eq. (27). Because $\gamma_{\overline{D}}$ and γ_C are derived from the combination of a displacement and rotation field, the associated compatibility conditions are altered, as is discussed in Section 7.

For the rotational strain, we can also derive similar definitions for α_D, $\alpha_{\overline{D}}$, α, and α_C and their related equations by replacing $\boldsymbol{u}_C$ with the contact rotation $\boldsymbol{w}_C$ in Eqs. (22)∼(30). We note that σ_C, μ_C, γ_C, and α_C are the basic tensorial form definitions for stress and strain in the discrete mechanics of granular assemblies.

Next, we consider the internal work done in a three-dimensional RVE. We might write the work as

$$W = \sum_C (\boldsymbol{f}_C \cdot \boldsymbol{u}_C + \boldsymbol{m}_C \cdot \boldsymbol{w}_C) = \sum_C 3V_C(\sigma_C \cdot \cdot \gamma_C + \mu_C \cdot \cdot \alpha_C), \tag{31}$$

but note that the Hill's condition (Hill, 1963)

$$W = V(\sigma \cdot \cdot \gamma + \mu \cdot \cdot \alpha) \tag{32}$$

does not hold for the stress and strain definitions in Eqs. (16) and (24). However, by combining Eqs. (7), (18) and (29), we can introduce a special double inner product $\circ\circ$ (with respect to quantities ξ_C and η_C for contacts C), having the property

$$\xi \circ \circ \eta = \frac{1}{V} \sum_C 3V_C(\xi_C \cdot \cdot \eta_C), \tag{33}$$

$$\xi = \frac{1}{V} \sum_C 3V_C \xi_C, \qquad \eta = \frac{1}{V} \sum_C 3V_C \eta_C, \tag{34}$$

so that Eq. (31) becomes

$$W = V(\sigma \circ \circ \gamma + \mu \circ \circ \alpha). \tag{35}$$

This equation is considered a modified form of Hill's condition for discrete, granular assemblies.

6. Tensorial form definition of discrete-mechanical quantities

As seen in the previous section, we need to introduce tensor definitions of discrete-mechanical quantities for the following reasons:

(1) To consider global, bulk mechanical properties, we must introduce averaging quantities that are tensorial.
(2) Because the mechanical quantities in continuum mechanics are tensorial, we must introduce definitions of discrete-mechanical quantities in a tensorial form in order to make comparisons between discrete- and continuum-mechanical systems.

Regarding the definitions of contact stress and contact strain explained in the previous section, we introduce the basic tensorial form definitions of discrete-mechanical quantities in Table 7. As is understood

Table 7
Basic tensorial form definitions of discrete-mechanical quantities

	Particle	Contact	Void	
2D				
Force Couple	$\begin{pmatrix} \boldsymbol{F}_P \\ M_P \end{pmatrix} = \frac{1}{S_P} \begin{pmatrix} \boldsymbol{f}_P \\ m_P \end{pmatrix}$,	$\begin{pmatrix} \sigma_C \\ \mu_C \end{pmatrix} = \frac{\boldsymbol{l}_C}{2S_C} \begin{pmatrix} \boldsymbol{f}_C \\ m_C \end{pmatrix}$,	$\begin{pmatrix} \varphi_D \\ \chi_D \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_D \\ m_D \end{pmatrix}$	
Rotation Displacement	$\begin{pmatrix} w_P \\ \boldsymbol{u}_P \end{pmatrix}$,	$\begin{pmatrix} \alpha_C \\ \gamma_C \end{pmatrix} = \frac{\boldsymbol{h}_C}{2S_C} \begin{pmatrix} w_C \\ \boldsymbol{u}_C \end{pmatrix}$,	$\begin{pmatrix} J_D \\ \boldsymbol{K}_D \end{pmatrix} = \frac{1}{S_D} \begin{pmatrix} w_D \\ \boldsymbol{u}_D \end{pmatrix}$	
			Dual-contact	Void cell
3D				
Force Couple	$\begin{pmatrix} \boldsymbol{F}_P \\ \boldsymbol{M}_P \end{pmatrix} = \frac{1}{V_P} \begin{pmatrix} \boldsymbol{f}_P \\ \boldsymbol{m}_P \end{pmatrix}$,	$\begin{pmatrix} \sigma_C \\ \mu_C \end{pmatrix} = \frac{\boldsymbol{l}_C}{3V_C} \begin{pmatrix} \boldsymbol{f}_C \\ \boldsymbol{m}_C \end{pmatrix}$,	$\begin{pmatrix} \varphi_D \\ \chi_D \end{pmatrix} = \frac{\boldsymbol{S}_D}{3V_D} \begin{pmatrix} \boldsymbol{f}_D \\ \boldsymbol{m}_D \end{pmatrix}$,	$\begin{pmatrix} \boldsymbol{f}_{\overline{D}} \\ \boldsymbol{m}_{\overline{D}} \end{pmatrix}$
Rotation Displacement	$\begin{pmatrix} \boldsymbol{w}_P \\ \boldsymbol{u}_P \end{pmatrix}$,	$\begin{pmatrix} \alpha_C \\ \gamma_C \end{pmatrix} = \frac{\boldsymbol{h}_C}{3V_C} \begin{pmatrix} \boldsymbol{w}_C \\ \boldsymbol{u}_C \end{pmatrix}$,	$\begin{pmatrix} \boldsymbol{J}_D \\ \boldsymbol{K}_D \end{pmatrix} = \frac{\boldsymbol{k}_D}{3V_D} \begin{pmatrix} \boldsymbol{w}_D \\ \boldsymbol{u}_D \end{pmatrix}$,	$\begin{pmatrix} \boldsymbol{X}_{\overline{D}} \\ \boldsymbol{Y}_{\overline{D}} \end{pmatrix} = \frac{1}{V_{\overline{D}}} \begin{pmatrix} \boldsymbol{w}_{\overline{D}} \\ \boldsymbol{m}_{\overline{D}} \end{pmatrix}$

from Table 7, the meanings of discrete-mechanical quantities defined for a dual-contact D, $(\boldsymbol{f}_D, \boldsymbol{m}_D)^{\mathrm{T}}$ and $(\boldsymbol{w}_D, \boldsymbol{u}_D)^{\mathrm{T}}$, are the stress function and incompatibility, respectively.

7. Conditions for stress and strain

In this section, we explain the conditions for stress and strain, comparing the related equations in continuum and discrete mechanics. In continuum mechanics, referring to Table 4, equilibrium in 3D is expressed as

$$Div\begin{pmatrix} \sigma \\ \mu \end{pmatrix} + \begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{M} \end{pmatrix} = 0, \tag{36}$$

where $(\boldsymbol{F}, \boldsymbol{M})^{\mathrm{T}}$ is the body force (density) given to the field beforehand. If we have a stress function $(\varphi, \chi)^{\mathrm{T}}$, which satisfies the relation

$$\begin{pmatrix} \sigma \\ \mu \end{pmatrix} = Rot\begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{37}$$

the body force must vanish (see Eq. (11)). In this case, the equilibrium condition of $(\sigma, \mu)^{\mathrm{T}}$ becomes

$$Div\begin{pmatrix} \sigma \\ \mu \end{pmatrix} = \begin{pmatrix} \nabla\cdot & 0 \\ \boldsymbol{I}\cdot\times & \nabla\cdot \end{pmatrix}\begin{pmatrix} \sigma \\ \mu \end{pmatrix} = 0. \tag{38}$$

In discrete mechanics, the equilibrium condition in 3D, which corresponds to Eq. (36), is written as (Tables 6 and 7)

$$-\widetilde{D}_{PC}\boldsymbol{h}_C \cdot \begin{pmatrix} \sigma_C \\ \mu_C \end{pmatrix} + V_P\begin{pmatrix} \boldsymbol{F}_P \\ \boldsymbol{M}_P \end{pmatrix} = 0, \tag{39}$$

where $(\boldsymbol{F}_P, \boldsymbol{M}_P)^{\mathrm{T}}$ is the average body force and couple (density) within a particle cell P, and V_P is the volume of particle cell P. For zero body force, the corresponding equations to Eqs. (37) and (38) are

$$\begin{pmatrix} \sigma_C \\ \mu_C \end{pmatrix} = \frac{\boldsymbol{l}_C}{3V_C} \widetilde{L}_{CD} \boldsymbol{k}_D \cdot \begin{pmatrix} \varphi_D \\ \chi_D \end{pmatrix}, \tag{40}$$

$$\widetilde{D}_{PC} \boldsymbol{h}_C \cdot \begin{pmatrix} \sigma_C \\ \mu_C \end{pmatrix} = \begin{pmatrix} D_{PC} & 0 \\ D_{PC} \boldsymbol{r}_{PC} \times & D_{PC} \end{pmatrix} \begin{pmatrix} \boldsymbol{h}_C \cdot \sigma_C \\ \boldsymbol{h}_C \cdot \mu_C \end{pmatrix} = 0, \tag{41}$$

where $\boldsymbol{h}_C$ and $\boldsymbol{k}_D$ are defined in Eqs. (6) and (9). Note that, in discrete mechanics, Eq. (13) are used in place of Eq. (11).

In continuum mechanics, the incompatibility $(\boldsymbol{J}, \boldsymbol{K})^{\mathrm{T}}$ is defined in the form

$$\begin{pmatrix} \boldsymbol{J} \\ \boldsymbol{K} \end{pmatrix} = Rot \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \tag{42}$$

and if the strains $(\alpha, \gamma)^{\mathrm{T}}$ are derived from the displacements and rotations $(\boldsymbol{w}, \boldsymbol{u})^{\mathrm{T}}$ by the relation

$$\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = Grad \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{u} \end{pmatrix}, \tag{43}$$

then the incompatibility must vanish (see Eq. (11)). In this case, we have the equation

$$Rot \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} \nabla \times & 0 \\ \boldsymbol{I} \times \times & \nabla \times \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = 0, \tag{44}$$

which is called the compatibility condition of strains $(\alpha, \gamma)^{\mathrm{T}}$.

In discrete mechanics, the corresponding equations to Eqs. (42)~(44) are written as

$$\begin{pmatrix} \boldsymbol{J}_D \\ \boldsymbol{K}_D \end{pmatrix} = \frac{\boldsymbol{k}_D}{3V_D} (\widetilde{L}_{DC})^{\mathrm{T}} \boldsymbol{l}_C \cdot \begin{pmatrix} \alpha_C \\ \gamma_C \end{pmatrix}, \tag{45}$$

$$\begin{pmatrix} \alpha_C \\ \gamma_C \end{pmatrix} = -\frac{\boldsymbol{h}_C}{3V_C} (\widetilde{D}_{CP})^{\mathrm{T}} \begin{pmatrix} \boldsymbol{w}_P \\ \boldsymbol{u}_P \end{pmatrix}, \tag{46}$$

$$(\widetilde{L}_{DC})^{\mathrm{T}} \boldsymbol{l}_C \cdot \begin{pmatrix} \alpha_C \\ \gamma_C \end{pmatrix} = \begin{pmatrix} L_{DC} & 0 \\ L_{DC} \boldsymbol{s}_{DC} \times & L_{DC} \end{pmatrix} \begin{pmatrix} \boldsymbol{l}_C \cdot \alpha_C \\ \boldsymbol{l}_C \cdot \gamma_C \end{pmatrix} = 0. \tag{47}$$

In 2D, the operators $D_{CP} \boldsymbol{r}_{PC} \times$ and $L_{DC} \boldsymbol{s}_{DC} \times$ in Eqs. (41) and (47) should be replaced by $D_{CP} \hat{\boldsymbol{r}}_{PC}$ and $L_{DC} \hat{\boldsymbol{s}}_{DC}$, respectively.

8. Discussions

For the strain in granular assemblies, Bagi (1996a) proposed a similar definition to that in this paper. Bagi's definition of strain is based on a discretization of the integral transform equation in continuum mechanics and has a form similar to the last right side of Eq. (27). One difference is that the so-called complementary area vector is used in place of the dual branch vector. As is explained by Satake (2002), this means that in Bagi's definition, the center of gravity of a void cell is used in place of the Dirichlet center, so that particle size is disregarded. It is noteworthy that, although any point may be used in place of the Dirichlet center, the use of the Dirichlet center makes the geometrical explanation most simple and clear, so that a systematic analysis, such as made in this paper, becomes easy both in 2D and 3D. Another difference between the strain definition in Eq. (27) and Bagi's definition is that Eq. (27) is based upon a contact displacement, that is the relative displacement of two points of a former contact point, whereas Bagi's definition is based upon the relative displacement of particle centers, as was explained after Eq. (30). The definition of strain in 2D using the center of gravity was also proposed by Kruyt and Rothenburg (1996).

As for internal work in a granular assembly, we can give the following extended expressions. If we assume that $(\boldsymbol{f}_C, \boldsymbol{m}_C)^{\mathrm{T}}$ and $(\boldsymbol{u}_D, \boldsymbol{w}_D)^{\mathrm{T}}$ are generated from $(\boldsymbol{f}_D, \boldsymbol{m}_D)^{\mathrm{T}}$ and $(\boldsymbol{u}_C, \boldsymbol{w}_C)^{\mathrm{T}}$ by the generating equations shown in Table 6, respectively, then we have

$$W = \sum_C (\boldsymbol{f}_C, \boldsymbol{m}_C) \cdot \begin{pmatrix} \mathbf{u}_C \\ \boldsymbol{w}_C \end{pmatrix} = \sum_C (\boldsymbol{f}_D, \boldsymbol{m}_D)(\widetilde{L}_{CD})^{\mathrm{T}} \cdot (\widetilde{L}_{DC})^{-1} \begin{pmatrix} \boldsymbol{u}_D \\ \boldsymbol{w}_D \end{pmatrix} = \sum_D (\boldsymbol{f}_D, \boldsymbol{m}_D) \cdot \begin{pmatrix} \boldsymbol{u}_D \\ \boldsymbol{w}_D \end{pmatrix}. \tag{48}$$

Using the relations shown in Table 7, we can write Eq. (48) as

$$\begin{aligned} W &= \sum_C 3V_C(\sigma_C, \mu_C) \cdot\cdot \begin{pmatrix} \gamma_C \\ \alpha_C \end{pmatrix} = \sum_C (\boldsymbol{f}_C, \boldsymbol{m}_C) \cdot \begin{pmatrix} \boldsymbol{u}_C \\ \boldsymbol{w}_C \end{pmatrix} = \sum_D (\boldsymbol{f}_D, \boldsymbol{m}_D) \cdot \begin{pmatrix} \boldsymbol{u}_D \\ \boldsymbol{w}_D \end{pmatrix} \\ &= \sum_D 3V_D(\varphi_D, \chi_D) \cdot\cdot \begin{pmatrix} \boldsymbol{K}_D \\ \boldsymbol{J}_D \end{pmatrix}. \end{aligned} \tag{49}$$

If we apply the special double product $\circ\circ$ defined by Eqs. (33) and (34) to dual-contacts D, we can write

$$W = V(\varphi \circ \circ \boldsymbol{K} + \chi \circ \circ \boldsymbol{J}), \tag{50}$$

where

$$\varphi = \frac{1}{V} \sum_D 3V_D \varphi_D, \qquad \boldsymbol{K} = \frac{1}{V} \sum_D 3V_D \boldsymbol{K}_D, \tag{51}$$

and similar equations for χ and $\boldsymbol{J}$.

Quite similarly, if we assume that $(\boldsymbol{f}_P, \boldsymbol{m}_P)^{\mathrm{T}}$ and $(\boldsymbol{u}_C, \boldsymbol{w}_C)^{\mathrm{T}}$ are generated from $(\boldsymbol{f}_C, \boldsymbol{m}_C)^{\mathrm{T}}$ and $(\boldsymbol{u}_P, \boldsymbol{w}_P)^{\mathrm{T}}$, respectively, we can write

$$W = V(\boldsymbol{F} \circ \circ \boldsymbol{u} + \boldsymbol{M} \circ \circ \boldsymbol{w}), \tag{52}$$

where

$$\boldsymbol{F} = \frac{1}{V} \sum_P V_P \boldsymbol{F}_P, \qquad \boldsymbol{u} = \frac{1}{V} \sum_P V_P \boldsymbol{u}_P, \tag{53}$$

and similar equations for $\boldsymbol{M}$ and $\boldsymbol{w}$. In this case, the special inner product $\circ\circ$ is applied for particles P. It is noted that equations similar to Eqs. (50) and (52) are derived in generalized continuum mechanics from the integral transform principles (Satake, 1971).

In continuum mechanics, the symmetric part of strain γ is written as ε and is simply called strain. In ordinary, classical continua we have

$$\boldsymbol{w} = \tfrac{1}{2} \nabla \times \boldsymbol{u}, \tag{54}$$

and accordingly ε coincides with γ. The compatibility condition of ε is written as

$$\nabla \nabla \times \times \varepsilon = 0. \tag{55}$$

In discrete mechanics, the symmetric parts of strains γ_C and γ, defined by Eqs. (29) and (27) respectively, are written as

$$\varepsilon_C = \frac{1}{2}(\gamma_C + \gamma_C^{\mathrm{T}}) = \frac{1}{6V_C}(\boldsymbol{h}_C \boldsymbol{u}_C + \boldsymbol{u}_C \boldsymbol{h}_C), \tag{56}$$

$$\varepsilon = \frac{1}{2}(\gamma + \gamma^{\mathrm{T}}) = \frac{1}{V} \sum_C 3V_C \varepsilon_C = \frac{1}{2V} \sum_C (\boldsymbol{h}_C \boldsymbol{u}_C + \boldsymbol{u}_C \boldsymbol{h}_C). \tag{57}$$

The compatibility condition of ε_C is written as

$$(L_{DC}\boldsymbol{l}_C\cdot)(L_{DC}\boldsymbol{l}_C\cdot)\varepsilon_C = -(L_{DC}\boldsymbol{l}_C\cdot)L_{DC}\boldsymbol{s}_{DC} \times \boldsymbol{w}_C. \tag{58}$$

It is noted here that, in the discrete mechanics of granular assemblies, the right side of Eq. (58) does not vanish, owing to the inhomogeneity of particle size and configuration in the assembly.

9. Concluding remarks

This paper proposed tensorial form definitions of discrete-mechanical quantities for granular assemblies using some new geometric tools based on the Dirichlet tessellation of an assembly. Although the tensorial form definition of stress has been already known, this paper adds similar tensorial form definitions for other mechanical quantities including the strain in discrete mechanics of granular assemblies. This paper explained how valuable such definitions are for considering the correspondence between discrete and continuum mechanics.

As the particles considered in this paper are limited to spheres or disks, a more general theory for granular assemblies with arbitrary shape (e.g. Bagi, 1996b) must become necessary. However, in the present stage, we put off this problem for a future work.

Appendix. Matrices D_{PC}, L_{DC}, and $C_{\overline{D}D}$

In this paper, we use the following matrices D_{PC}, L_{DC}, and $C_{\overline{D}D}$ to express the topologic correspondence of the three or four elements of a granular assembly; P, C, and D in 2D; P, C, D, and $\overline{D}$ in 3D (see Table 1):

The *incidence matrix* D_{PC} for a particle graph (oriented graph) is defined as

$$D_{PC} = \begin{cases} 1: & \text{if branch vector } \boldsymbol{l}_C \text{ is incident at particle center } P \text{ and is oriented away from } P, \\ -1: & \text{if branch vector } \boldsymbol{l}_C \text{ is incident at particle center } P \text{ and is oriented toward } P, \\ 0: & \text{otherwise,} \end{cases} \tag{A.1}$$

where the direction of branch vector $\boldsymbol{l}_C$ is arbitrarily given beforehand.

The *loop matrix* L_{DC} for a triangular loop (void cell in 2D, or dual-contact in 3D) D is defined as

$$L_{DC} = \begin{cases} 1: & \text{if branch vector } \boldsymbol{l}_C \text{ is included in loop } D \text{ and the orientation of } \boldsymbol{l}_C \\ & \quad \text{and that of loop } D \text{ coincide,} \\ -1: & \text{if branch vector } \boldsymbol{l}_C \text{ is included in loop } D \text{ and the orientation of } \boldsymbol{l}_C \\ & \quad \text{and that of loop } D \text{ do not coincide,} \\ 0: & \text{otherwise,} \end{cases} \tag{A.2}$$

where the orientation (clockwise or counter-clockwise) of loop D is fixed as clockwise in 2D and is arbitrarily given beforehand in 3D.

In 3D, we need the cell matrix $C_{\overline{D}D}$ for a void cell $\overline{D}$ defined as

$$C_{\overline{D}D} = \begin{cases} 1: & \text{orientation of loop (dual-contact) } D \text{ is counter-clockwise as a face of void cell } \overline{D}, \\ -1: & \text{orientation of loop (dual-contact) } D \text{ is clockwise as a face of void cell } \overline{D}, \\ 0: & \text{otherwise.} \end{cases} \tag{A.3}$$

As is known in the graph theory, the above matrices satisfy the following identities:

$$\left.\begin{array}{ll} D_{PC}L_{CD} = 0, & L_{CD}C_{D\overline{D}} = 0, \\ C_{\overline{D}D}L_{DC} = 0, & L_{DC}D_{CP} = 0, \end{array}\right\} \tag{A.4}$$

where D_{CP}, L_{CD} and $C_{D\overline{D}}$ are the transpose of D_{PC}, L_{DC} and $C_{\overline{D}D}$, respectively.

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